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ON THE GROUP OF AUTOMORPHISMS OF A SURFACE $x^n y = P(z)$

BY

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To the memory of Boris Moishezon

ABSTRACT

In this note the AK invariant of a surface in $\mathbb{C}^3$ which is given by $x^n y = P(z)$ where $n > 1$ and $\deg(P) = d > 1$ is computed. Then this information is used to find the group of automorphisms of this surface and the isomorphism classes of such surfaces.

Introduction

Let A be a (not necessarily commutative) algebra over a field F. Typically a description of the group of automorphisms of A, one of the most important characteristics of an algebra, is a difficult problem. Recall that even for the most "symmetric" algebras, i.e. for algebras of polynomials and for free associative algebras, we know the answers only when the number of generators is less than three (see [C]). In the commutative case the geometric counterpart of this question is a description of the automorphisms of affine algebraic varieties. So polynomial algebras correspond to affine spaces F^n. Though the question on algebraic automorphisms of the plane was settled long ago (see [J], [vdK]) and, arguably,

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the most important question of three-dimensional affine algebraic geometry is to describe the group of automorphisms of F^3, this is a wide open problem.

Even for surfaces we do not know too much since we are lacking any general technique to attack the question.

In this note we describe the groups of automorphisms of the hypersurfaces in $\mathbb{C}^3$ which are given by $x^n y = P(z)$ where $n > 1$ and the degree d of P is also larger than 1. The case $n = 1$ was considered in [DG1], [DG2], and [ML1], and if $d = 1$ the surface is a plane (see [J]).

In [B] some information on the groups of automorphisms of a more general class of surfaces is obtained. In [W] techniques of algebraic geometry are used to obtain similar results for a class of surfaces which slightly overlaps with the class considered in our paper.*

This is an interesting class of surfaces which has appeared before in connection with the generalized Zariski cancellation question:

Let V and W be affine varieties over a field F. Is it true that $V \times F^k \simeq W \times F^k$ implies $V \simeq W$?

If V and W are curves the answer is positive (see [AEH]). But even for surfaces this is not the case. The surfaces $x^n y = z^2 - 1$ for $n = 1$ and $n = 2$ provide a negative answer as was shown by Danielewski [Da]. Danielewski also established that the cylinders over these surfaces are isomorphic for all n. Later Fieseler [F] proved that the surfaces with different n are all non-isomorphic by computing the first homology group at infinity.

In [W] a wider class of surfaces which are pairwise non-isomorphic while all the cylinders are isomorphic is presented.

Following ideas of Dixmier [Di] and Rentschler [R] the author introduced a ring invariant AK which distinguishes from $\mathbb{C}^3$ the Koras–Russell threefolds (see [ML2] and [KML]). This was the crucial step which allowed Koras and Russell to finish their consideration of $\mathbb{C}^*$-actions on $\mathbb{C}^3$ and to show that these actions are linearizable (see [KKMLR]).

Here we compute the AK invariant for the surfaces $x^n y = P(z)$ in $\mathbb{C}^3$. It turns out that when $n > 1$ and the degree of P is also larger than 1, the AK invariant for these surfaces is $\mathbb{C}[x]$. After that it is rather easy to describe the automorphism groups of these surfaces and to classify them up to isomorphism. This generalizes the result of Fieseler since the surfaces in our class are not necessarily normal.

* This paper was brought to the author's attention by one of the referees.

Definitions

All algebras in this paper have characteristic zero.

Let A be a (not necessarily commutative) algebra over a field F. Let $\mathrm{Der}(A)$ be the Lie algebra of all F-derivations on A, i.e., F-linear homomorphisms on A which satisfy the Leibniz rule ($\partial(ab) = \partial(a)b + a\partial(b)$). For any derivation ∂ the set A^∂ denotes the kernel of ∂. It is easy to check that A^∂ is a subalgebra. This subalgebra is usually called the ring of ∂-constants. Let $\mathrm{LND}(A) \subset \mathrm{Der}(A)$ be the set of all locally nilpotent derivations of A. A derivation ∂ is locally nilpotent if for each $a \in A$ there exists a natural number $n = n(a)$ for which $\partial^n(a) = 0$. In the case when A is the ring of regular functions on an algebraic variety, a locally nilpotent derivation gives rise via exponentiation to a $\mathbb{C}^+$-action on the variety (e.g. see [Sn]).

AK INVARIANT. The intersection of the rings of constants for all locally nilpotent derivations will be called the ring of absolute constants and denoted by $\mathrm{AK}(A)$.

The set $\mathrm{LND}(A)$ and the ring $\mathrm{AK}(A)$ play obvious roles in the investigation of the automorphisms of A: any automorphism induces an automorphism of $\mathrm{AK}(A)$ and acts on $\mathrm{LND}(A)$ by conjugation. So it is useful in this context to describe them if possible. They may also be helpful when investigating isomorphisms between rings.

General facts about derivations

LEMMA 1: *Let A be a commutative domain and let ∂ be a derivation of A. If $\partial(g) \neq 0$ then g is algebraically independent over A^∂.*

Proof: Assume that $R(g) = 0$ where $R(x) \in A^\partial[x]$ and has minimal possible degree. Then $0 = \partial(R(g)) = R'(g)\partial(g)$ where R' is the ordinary derivative. So $R'(g) = 0$, which is a contradiction.

LEMMA 2: *Let $\partial \in \mathrm{Der}(A)$ where A is a subring of $F(x_1, \ldots, x_n)$. If the transcendence degree of A^∂ is $n-1$ and $f_1, \ldots, f_{n-1}$ is a transcendence basis of A^∂ then there exists an $h \in F(x_1, \ldots, x_n)$ so that $\partial(a) = hJ(f_1, \ldots, f_{n-1}, a)$ for every $a \in A$. Here $J(f_1, \ldots, a)$ is the Jacobian relative to $x_1, \ldots, x_n$.*

Proof: Any derivation of $F(x_1, \ldots, x_n)$ is completely determined by its values on any n algebraically independent elements. Let $\epsilon(a) = J(f_1, \ldots, f_{n-1}, a)$. Then $\epsilon(f_i) = 0$ for $i = 1, \ldots, n-1$. Let us take any $g \in A$ for which $\partial(g) \neq 0$.

By Lemma 1, $\epsilon(g) \neq 0$. Since $h\epsilon$ is a derivation of $F(x_1, \ldots, x_n)$ for any $h \in F(x_1, \ldots, x_n)$ it is sufficient to determine h by $h = \partial(g)(\epsilon(g))^{-1}$.

With the help of a locally nilpotent derivation ∂ acting on a ring A one can define a function $\deg_\partial$ by $\deg_\partial(f) = \max(n | \partial^n(f) \neq 0)$ if $f \in A$ is not zero and $\deg_\partial(0) = -\infty$.

LEMMA 3: *If A is a domain and $\partial \in \mathrm{LND}(A)$ then*

(i) $\deg_\partial$ *is a degree function, i.e.*

$$\deg_\partial(a+b) \leq \max(\deg_\partial(a), \deg_\partial(b)) \quad \text{and} \quad \deg_\partial(ab) = \deg_\partial(a) + \deg_\partial(b).$$

(ii) *The ring A^∂ is "factorially closed", i.e. if $a, b \in A$ where a and b are non-zero elements and $ab \in A^\partial$ then $a, b \in A^\partial$.*

The proof of this lemma is an easy exercise. Or see [FLN].

Let A be a ring with an ascending $\mathbb{Z}$-filtration $\{A_i\}$ for which $\bigcap_{i=-\infty}^{\infty} A_i = 0$ and let ∂ be a derivation on A for which $\partial(A_i) \subset A_{i+k}$ for a fixed k and all i. Let $\mathrm{Gr}(A) = \bigoplus A_i/A_{i-1}$ be the corresponding graded ring and let $h \in A_i/A_{i-1}$. Let us write $h = a + A_{i-1}$ where $a \in A_i$. We can define a homomorphism ∂_1 on $\mathrm{Gr}(A)$ which acts on h by $\partial_1(h) = \partial(a) + A_{i+k-1} \in A_{i+k}/A_{i+k-1}$ and then extend ∂_1 on $\mathrm{Gr}(A)$ by linearity. It is clear that ∂_1 is a derivation of $\mathrm{Gr}(A)$. If k is chosen minimal (and $k \neq -\infty$) then $\partial_1 \neq 0$.

By definition ∂_1 sends a homogeneous component A_i/A_{i-1} into the homogeneous component A_{i+k}/A_{i+k-1}. So we can call ∂_1 a homogeneous derivation of degree k.

Let us also define a non-linear mapping gr from A to $\mathrm{Gr}(A)$ by $\mathrm{gr}(a) = a + A_{i-1}$ if $a \in A_i \setminus A_{i-1}$, $\mathrm{gr}(0) = 0$. Then $\partial_1(\mathrm{gr}(a))$ is either $\mathrm{gr}(\partial(a))$ or 0 by definition of ∂_1.

The proof of the following lemma is straightforward. (Or see Lemma 4 from [ML2].)

LEMMA 4: *If ∂ is a locally nilpotent derivation on A then ∂_1 is a locally nilpotent derivation on* $\mathrm{Gr}(A)$.

Locally nilpotent derivations of S

Let $S = \mathbb{C}[X, Y, Z]/(Q)$ where $Q = X^nY - P(Z)$, $n > 1$, and $P(Z)$ is a polynomial with degree $d > 1$. Let x, y, z be the images of X, Y, Z in S.

We'll describe here all locally nilpotent derivations of the ring S and show that $\mathrm{AK}(S) = \mathbb{C}[x]$.

It is sometimes possible to obtain information on a locally nilpotent derivation by transition to the corresponding homogeneous locally nilpotent derivations induced by different filtrations. This approach will be used in this section.

Let us consider S as a subring of the ring $T = \mathbb{C}[x, x^{-1}, z]$. So $y = x^{-n}P(z)$. We are going to use filtrations of S which are induced by so-called weight filtrations of T. To define such a filtration on T it is sufficient to give any real weights μ and ν to x and z. Then the weight of a monomial $x^i z^j$ is $i\mu + j\nu$ and the weight of an element p from T is the maximal weight of monomials of p.

Since any element of T can be presented as the sum of homogeneous elements this weight function gives T the structure of a graded algebra, so $\mathrm{Gr}(T) \simeq T$. Of course if we want a $\mathbb{Z}$-graded algebra we should take μ and ν from $\mathbb{Z}$. The filtration on T is given, as usual, by taking T_n to be the linear span of $\{x^i z^j | i\mu + j\nu \leq n\}$ and $S_n = T_n \cap S$.

Let us identify $\mathrm{Gr}(T)$ and T and denote $\mathrm{gr}(x)$ by x and $\mathrm{gr}(z)$ by z. If the weight ν of z is positive then $\mathrm{gr}(y) = x^{-n}z^d$ and $\mathrm{Gr}(s)$ is generated by x, z, and $x^{-n}z^d$. Indeed, any element of S can be written as a polynomial in x, y, and z. Because of the relation $x^n y = P(z)$ we can rewrite it as a sum of monomials $x^i z^j y^k$ such that $i < n$ whenever $k > 0$. It is easy to see that the monomials $x^i z^j (x^{-n}z^d)^k$ which are obtained from these monomials by replacing y with $\mathrm{gr}(y)$ are linearly independent. So the algebra $\mathrm{Gr}(S)$ is the linear span of these monomials and is generated by x, z, and $\mathrm{gr}(y)$.

It is also possible to extend the weight to the rational functions $\mathbb{C}(x, z)$ by defining the weight of pq^{-1} as the difference of the weights of p and q. The associated graded algebra $\mathrm{Gr}(\mathbb{C}(x, z))$ is isomorphic to the subalgebra of $\mathbb{C}(x, z)$ consisting of fractions with homogeneous denominators.

Let ∂ be a non-zero locally nilpotent derivation of S and let $f \in S^{\partial} \setminus \mathbb{C}$. Such an f exists since by [DF] the kernel of ∂ has transcendence degree one. Then $\partial(g) = hJ(f, g)$ where J is the Jacobian relative to x and z, and $h \in \mathbb{C}(x, z)$ (see Lemma 2).

Lemma 5: $f \in \mathbb{C}[x, z]$.

Proof: Let us call the order of an element $s \in S$ the smallest degree of x appearing in the monomials of s. Let us take weights $\mu = -N$, $\nu = 1$ where N is a positive integer which is sufficiently large to make the leading forms $\mathrm{gr}(f) \in \mathrm{Gr}(S)$ and $\mathrm{gr}(h) \in \mathrm{Gr}(\mathbb{C}(x, z))$ monomials with the same powers of x as the orders of f and h, respectively.

Let ∂_1 be the locally nilpotent derivation of $\mathrm{Gr}(S)$ which is induced by ∂ for $k = \deg(h) + \deg(f) - \deg(x) - \deg(z)$ (see Lemma 4).* It is clear that $\partial(S_i) \subset S_{i+k}$. We can check that $\partial_1(\mathrm{gr}(g)) = \mathrm{gr}(h)J(\mathrm{gr}(f), \mathrm{gr}(g))$ for every $g \in S$. Indeed, $\partial_1(\mathrm{gr}(g)) = \mathrm{gr}(\partial(g))$ if $\deg(\partial(g)) = \deg(g) + k$ and $\partial_1(\mathrm{gr}(g)) = 0$ if $\deg(\partial(g)) < \deg(g) + k$. But $\mathrm{gr}(\partial(g)) = \mathrm{gr}(hJ(f,g))$ and by definition of the Jacobian $\mathrm{gr}(J(f,g)) = J(\mathrm{gr}(f), \mathrm{gr}(g))$ if the latter is non-zero. In this case $\deg(J(f,g)) = \deg(f) + \deg(g) - \deg(x) - \deg(z)$ and $\partial_1(\mathrm{gr}(g)) = \mathrm{gr}(h)J(\mathrm{gr}(f), \mathrm{gr}(g))$ by definition of k. If instead $J(\mathrm{gr}(f), (\mathrm{gr}(g)) = 0$, then $\deg(J(f,g)) < \deg(f) + \deg(g) - \deg(x) - \deg(z)$ and again by definition of k we have $\partial_1(\mathrm{gr}(g)) = 0$. In either case $\partial_1(\mathrm{gr}(g)) = \mathrm{gr}(h)J(\mathrm{gr}(f), \mathrm{gr}(g))$. The derivation ∂_1 is identically zero if and only if $\mathrm{gr}(f) \in \mathbb{C}$.

Let us assume that $f \notin \mathbb{C}[x, z]$. As mentioned above any element of S can be presented as a Laurent polynomial from $\mathbb{C}[x, x^{-1}, z]$. So f contains monomials with negative degree in x. Since we choose the filtration to pick the monomial with the smallest x-degree possible we have that $\mathrm{gr}(f)$ is a monomial with negative power of x.

This means that $\mathrm{gr}(f)$ can be written as a monomial $x^a z^b \,\mathrm{gr}(y)^c$ where a, b, and c are non-negative integers. So $\mathrm{gr}(f)$ is divisible by $\mathrm{gr}(y)$ in $\mathrm{Gr}(S)$ because negative powers of x occur only when $c > 0$.

Since $\mathrm{gr}(f)$ is a ∂_1-constant and a non-zero product of elements from $\mathrm{Gr}(S)$ is a ∂_1-constant only if all factors are constants (see Lemma 3), this means that $\mathrm{gr}(y)$ is a constant. Since ∂_1 is not the zero derivation, $\mathrm{gr}(f) = \mathrm{gr}(y)^c$ where $c > 0$ (otherwise ∂_1 will be zero on two algebraically independent elements of $\mathrm{Gr}(S)$ and therefore zero by the proof of Lemma 2). Now, $\partial_1(\mathrm{gr}(g)) = \mathrm{gr}(h)J(\mathrm{gr}(y)^c, \mathrm{gr}(g)) = c\,\mathrm{gr}(y)^{c-1}\,\mathrm{gr}(h)J(\mathrm{gr}(y), \mathrm{gr}(g))$ is a locally nilpotent derivation. Let us denote $\mathrm{gr}(y)^{c-1}\,\mathrm{gr}(h)$ by $c_1 x^a z^b$. So forgetting about the coefficients, $\partial_2(g) = x^a z^b J(\mathrm{gr}(y), g)$ defines a locally nilpotent derivation on $\mathrm{Gr}(S)$.

Let us use the conditions that $\partial_2(x)$ and $\partial_2(z)$ are some elements of $\mathrm{Gr}(S)$. These conditions can be written as

$$(a, b) + (-n, d-1) = \alpha_x(-n, d) + (\beta_x, \gamma_x)$$

and

$$(a, b) + (-n-1, d) = \alpha_z(-n, d) + (\beta_z, \gamma_z),$$

respectively, where all scalars in the right sides are non-negative integers. They

* Here deg denotes the weight function induced by $\deg(x) = -N$ and $\deg(z) = 1$.

may be rewritten as

$$a + (\alpha_x - 1)n \geq 0, \quad b - 1 - (\alpha_x - 1)d \geq 0,$$
$$a - 1 + (\alpha_z - 1)n \geq 0, \quad b - (\alpha_z - 1)d \geq 0$$

or

$$\frac{b-1}{d} \geq (\alpha_x - 1) \geq \frac{-a}{n}, \quad \frac{b}{d} \geq (\alpha_z - 1) \geq \frac{1-a}{n}.$$

Let us use now the condition that ∂_2 is locally nilpotent. Since $\partial_2(x^i z^j) = -(nj+di)x^{i+a-n-1}z^{j+b+d-1}$ it means that $n[j+k(b+d-1)]+d[i+k(a-n-1)] = 0$ for some non-negative integer k for all $x^i z^j \in \mathrm{Gr}(S)$. In particular,

$$\frac{nj+di}{n+d-nb-da} \in \mathbb{N} \quad \text{for any } i, j \in \mathbb{N}.$$

Let us denote $n(1-b)+d(1-a)$ by Δ. We should have therefore $\Delta > 0$ (and Δ divides n and d). Hence

$$\frac{b}{d} + \frac{a}{n} = \frac{bn+ad}{dn} = \frac{n+d-\Delta}{nd} < \frac{n+d}{nd} \leq 1$$

since we assumed that $n > 1$ and $d > 1$. So the integers $\alpha_x - 1$ and $\alpha_z - 1$ must be equal since they both belong to the interval $[-a/n, b/d]$. But then $(b-1)/d \geq (1-a)/n$, which means that $0 \geq \Delta$. We have reached a contradiction which proves the lemma.

So we have proved that $\partial(g)$ can be written as $\partial(g) = hJ(f,g)$ where $f \in \mathbb{C}[x,z]$. Let us refine it further.

LEMMA 6: $f \in \mathbb{C}[x]$.

Proof: Let us take $\mu = 1$, $\nu = N$ where N is sufficiently large to make $\mathrm{gr}(f)$ and $\mathrm{gr}(h)$ monomials which have the same degrees in z as do f and h, respectively. Let ∂_1 be the locally nilpotent derivation of $\mathrm{Gr}(S)$ which is induced by ∂ (see Lemma 4). As in the proof of Lemma 5, $\partial_1(\mathrm{gr}(g)) = \mathrm{gr}(h)J(\mathrm{gr}(f), \mathrm{gr}(g))$ where $\mathrm{gr}(f)$ is divisible by z in $\mathrm{Gr}(S)$ if $f \notin \mathbb{C}[x]$. So z is a constant if $f \notin \mathbb{C}[x]$. But then z^d is also a constant and it is divisible by $\mathrm{gr}(y)$ in $\mathrm{Gr}(S)$. So both $\mathrm{gr}(y) = x^{-n}z^d$ and z are constants and this implies that ∂_1 is identically zero.

Therefore $f \in \mathbb{C}[x]$.

From these two lemmas we see that some polynomial in x is a constant of a non-zero locally nilpotent derivation. From Lemma 1 (or from the representation of a derivation as a Jacobian), we see then that x is a constant of any non-zero locally nilpotent derivation, and we can write ∂ as $\partial(g) = hJ(x,g)$, i.e. $\partial = h\partial/\partial z$. If $\deg_z(h) > 0$ then $\partial^n(z) \neq 0$ for any n. So $h \in \mathbb{C}(x)$. Of course $h = \partial(z) \in S$. Since $S \cap \mathbb{C}(x) = \mathbb{C}[x]$ we see that $h \in \mathbb{C}[x]$.

Let us take now the same filtration as in Lemma 5. Then $\mathrm{Gr}(S)$ is generated by x, z, and $\mathrm{gr}(y) = x^{-n}z^d$. Now, $\partial(y) = hx^{-n}P'(z) \in S$ (here $P'(z)$ is the ordinary derivative relative to z) and $\mathrm{gr}(\partial(y)) = d\,\mathrm{gr}(h)x^{-n}z^{d-1} \in \mathrm{Gr}(S)$. If the order of h is k then $x^{k-n}z^{d-1} \in \mathrm{Gr}(S)$. So $k - n \geq 0$, which means that h is divisible by x^n in $\mathbb{C}[x]$.

Therefore $h = x^n h_1$ where $h_1 \in \mathbb{C}[x]$.

PROPOSITION: (i) *A derivation ∂ of S is locally nilpotent if and only if $\partial(g) = x^n h_1(x)\partial/\partial z$ where $h_1 \in \mathbb{C}[x]$.*

(ii) $\mathrm{AK}(S) = \mathbb{C}[x]$.

Proof: We need only check that $\partial = x^n\partial/\partial z \in \mathrm{LND}(S)$. This is so because ∂ is even locally nilpotent on T and $\partial(S) \subset S$.

Let us use this information for a description of the automorphisms.

Automorphisms of S

Some of the automorphisms of S are quite evident. First of all we have a $\mathbb{C}^*$-action $\lambda(x,y,z) = (\lambda x, \lambda^{-n}y, z)$. Secondly, since the exponent of a locally nilpotent derivation gives an automorphism (see [Sn]) we also have an additive $\mathbb{C}[x]$ action $h(x)(x,y,z) = (x, y + (P(z+hx^n) - P(z))x^{-n}, z + hx^n))$. It turns out that for a typical $P(z)$ these automorphisms generate the whole group $\mathrm{Aut}(S)$.

Let us make a linear substitution in z so that $P(z)$ will become a monic polynomial with zero coefficient of z^{d-1}.

LEMMA 7: *Let $\alpha \in \mathrm{Aut}(S)$. Then $\alpha(x) = c_1 x$ and $\alpha(z) = c_2 z + b(x)$ where $c_1, c_2 \in \mathbb{C}^*$, $b(x) \in \mathbb{C}[x]$, $b(x) \equiv 0 \pmod{x^n}$, and $P(c_2 z) = c_2^d P(z)$.*

Proof: Since α induces an automorphism of $\mathrm{AK}(S) = \mathbb{C}[x]$ we see that $\alpha(x) = c_1 x + b_1$ where $c_1 \in \mathbb{C}^*$ and $b_1 \in \mathbb{C}$. Next $\partial^2(z) = 0$ for any $\partial \in \mathrm{LND}(S)$. Therefore $\partial^2(\alpha(z)) = 0$ for any $\partial \in \mathrm{LND}(S)$ and $\alpha(z) = c_2 z + b$ where $c_2, b \in \mathbb{C}[x]$. Since α is invertible we see that $c_2 \in \mathbb{C}^*$.

Let $\epsilon = x^n\partial/\partial z$. Then $\alpha^{-1}\epsilon\alpha$ is also a locally nilpotent derivation of S. Now, $\alpha^{-1}\epsilon\alpha(z) = c_1^{-n}c_2(x - b_1)^n$. But $\partial(z)$ is divisible by x^n for any locally nilpotent ∂. So $b_1 = 0$.

Next, $\alpha(x^n y) = c_1^n x^n \alpha(y) = P(c_2 z + b) = c_2^d P(z) + \Delta(x,z)$ where $\Delta(x,z) \in \mathbb{C}[x,z]$ and $\deg_z(\Delta) < d$. So $c_1^n\alpha(y) = x^{-n}(c_2^d P(z) + \Delta(x,z)) = c_2^d y + \Delta(x,z)x^{-n}$.

Since $\Delta(x,z)x^{-n} \in S = \mathbb{C}[x,y,z]$ and $\deg_z(\Delta) < d$ this means that $\Delta \equiv 0 \pmod{x^n}$. Since $\Delta = P(c_2z+b) - c_2^dP(z) = dc_2^{d-1}z^{d-1}b + \delta$ where $\deg_z \delta < d-1$ we see that $b \equiv 0 \pmod{x^n}$. Therefore $P(c_2z+b) \equiv P(c_2z) \pmod{x^n}$ and $\Delta \equiv P(c_2z) - c_2^dP(z) \equiv 0 \pmod{x^n}$, which is possible only if $P(c_2z) - c_2^dP(z) = 0$.

Now we are ready to check the following

THEOREM 1: *The group* $\mathrm{Aut}(S)$ *is generated by the following automorphisms.*

(a) $H(x) = \lambda x$, $H(y) = \lambda^{-n}y$, $H(z) = z$ *where* $\lambda \in \mathbb{C}^*$.

(b) $T(x) = x$, $T(y) = y + [P(z + x^nf(x)) - P(z)]x^{-n}$, $T(z) = z + x^nf(x)$, *where* $f(x) \in \mathbb{C}[x]$.

(c) *If* $P(z) = z^d$ *then the automorphisms* $R(x) = x$, $R(y) = \lambda^dy$, $R(z) = \lambda z$ *where* $\lambda \in \mathbb{C}^*$ *should be added.*

(d) *If* $P(z) = z^ip(z^m)$ *then the automorphisms* $S(x) = x$, $S(y) = \mu^dy$, $S(z) = \mu z$ *where* $\mu \in \mathbb{C}$ *and* $\mu^m = 1$ *should be added.*

Proof: It is clear that all of these transformations are automorphisms. It is also clear from Lemma 7 that any automorphism is a composition of an automorphism H, an automorphism T, and an automorphism α for which $\alpha(x) = x$ and $\alpha(z) = cz$. Cases (c) and (d) describe all polynomials P for which $P(cz) - c^dP(z) = 0$ with $c \neq 1$ is possible. (In all other cases α is the identity automorphism.)

Remark: The triangular automorphisms T form a normal subgroup which is isomorphic to the additive group of $\mathbb{C}[x]$ and the group $\mathrm{Aut}(S)$ is a semidirect product of T and L where L is the subgroup of linear automorphisms generated by the automorphisms from (a), (c), and (d).

In the general case when (d) is not satisfied, L is isomorphic to $\mathbb{C}^*$. If (d) is satisfied but not (c), then L is isomorphic to the direct product of $\mathbb{C}^*$ and a cyclic group C_m. Finally, if (c) is satisfied, then L is isomorphic to the direct product of two copies of $\mathbb{C}^*$.

The group $\mathrm{Aut}(S)$ is a metabelian group.

Isomorphisms of S

Let S_1 and S_2 be two algebras which correspond to $Q_1 = X_1^{n_1}Y_1 - P_1(Z_1)$ and $Q_2 = X_2^{n_2}Y_2 - P_2(Z_2)$ where n_1, n_2, d_1, and d_2 are all larger than 1. We also assume that P_i are monic polynomials with zero coefficients of z^{d_i-1}.

THEOREM 2: *$S_1 \cong S_2$ if and only if $n_1 = n_2 = n$, $d_1 = d_2 = d$, and $P_2(z) = \lambda^{-d}P_1(\lambda z)$ where $\lambda \in \mathbb{C}^*$.*

Proof: Let α be an isomorphism of these algebras. We know that $\mathrm{AK}(S_i) = \mathbb{C}[x_i]$ and that $\mathrm{LND}(S_i) = x_i^{n_i}\mathbb{C}[x_i]\partial/\partial z_i$. So as in Lemma 7 we can conclude that $\alpha(x_1) = c_1x_2$ and $\alpha(z_1) = c_2z_2 + b(x_2)$ where $c_1, c_2 \in \mathbb{C}^*$, $b(x) \in \mathbb{C}[x]$. We may even assume using Theorem 1 that $c_1 = 1$. We may also assume without loss of generality that $d_1 \leq d_2$ because we can switch S_1 and S_2.

Let us assume that $d_1 < d_2$. Then $\alpha(y_1) = x_2^{-n_1}P_1(c_2z_2 + b) \notin S_2$ since the elements from S_2 with negative power of x_2 should contain z_2 in the power d_2 at least. So $d_1 = d_2 = d$. Similarly, the assumption that $n_1 > n_2$ brings us to a contradiction. Therefore $n_1 = n_2 = n$.

Now we can see that $b(x) \equiv 0 \pmod{x^n}$, so using Theorem 1 again we may assume that $\alpha(z_1) = cz_2$. Finally, $P_1(cz) = c^dP_2(z)$ as in Lemma 7. So $P_2(z) = c^{-d}P_1(cz)$.

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